A note on geometric 3-hypergraphs

Andrew Suk*

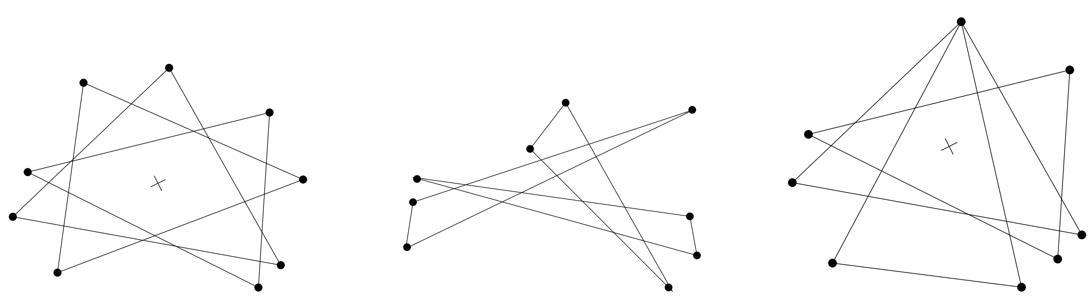
November 10, 2011

Abstract

In this note, we prove several Turán-type results on geometric hypergraphs. The two main theorems are 1) Every n-vertex geometric 3-hypergraph in the plane with no three strongly crossing edges has at most $O(n^2)$ edges, 2) Every n-vertex geometric 3-hypergraph in 3-space with no two disjoint edges has at most $O(n^2)$ edges. These results support two conjectures that were raised by Dey and Pach, and by Akiyama and Alon.

1 Introduction

A *geometric r-hypergraph H in d-space* is a pair (V, E), where V is a set of points in general position in Euclidean d-space, and E is a set of closed $(r-1)$-dimensional simplices (edges) induced by some r-tuple of V. The sets V and E are called the *vertex set* and *edge set* of H, respectively. Two edges in H are *crossing* if they are vertex disjoint and have a point in common. Notice that if k edges are pairwise crossing, it does not imply that they all have a point in common. Hence we say that H contains *k strongly crossing edges* if H contains k vertex disjoint edges that all share a point in common. See Figure 1.



(a) Three strongly crossing edges.

(b) Three pairwise crossing edges with an empty intersection.

(c) Three edges not strongly crossing since two share a vertex.

Figure 1: Three edges of a geometric 3-hypergraph in the plane.

A direct application of the colored Tverberg theorem (see [3],[20]) gives

*Courant Institute, New York and EPFL, Lausanne. Email: `suk@cims.nyu.edu`. The author gratefully acknowledges the support from the Swiss National Science Foundation, Grant No. 200021-125287/1.

Theorem 1.1. *Let $ex_d(SC_k^{d+1}, n)$ denote the maximum number of edges an n-vertex geometric $(d+1)$-hypergraph in d-space has with no k strongly crossing edges. Then*

$$ex_d(SC_k^{d+1}, n) = O\left(n^{d+1-\frac{1}{(2k-1)^d}}\right).$$

Dey and Pach [5] showed that $ex_d(SC_2^{d+1}, n) = \Theta(n^d)$, and conjectured $ex_d(SC_k^{d+1}, n) = \Theta(n^d)$ for every fixed d and k. The lower bound can easily be seen by taking all edges with a vertex in common. The main motivation for their conjecture is for deriving upper bounds on the maximum number of *k-sets* of an n-point set in $\mathbb{R}^d$. See [12] for more details. In this note, we settle the Dey-Pach conjecture for geometric 3-hypergraphs in the plane with no three strongly crossing edges, and improve the upper bound of $ex_2(SC_k^3, n)$.

Theorem 1.2. $ex_2(SC_3^3, n) = \Theta(n^2)$.

Theorem 1.3. *For fixed $k \geq 4$, $ex_2(SC_k^3, n) \leq O(n^{3-\frac{1}{k}})$.*

As a related result, Akiyama and Alon [2] used the Borsuk-Ullam Theorem [4] to show the following.

Theorem 1.4. *Let $ex_d(D_k^d, n)$ denote the maximum edges that an n-vertex geometric d-hypergraph in d-space has with no k pairwise disjoint edges. Then*

$$ex_d(D_k^d, n) \leq n^{d-(1/k)^{d-1}}.$$

They conjecture that for every fixed d and k, $ex_d(D_k^d, n) = \Theta(n^{d-1})$. Again the lower bound can easily be seen by taking all edges with a vertex in common. Pach and Törőcsik [15] showed that $ex_2(D_k^2, n) = O(k^4 n)$, which was later improved to $O(k^2 n)$ by Tóth [17]. Here we settle the Akiyama-Alon conjecture for geometric 3-hypergraphs in 3-space with no two disjoint edges.

Theorem 1.5. $ex_3(D_2^3, n) = \Theta(n^2)$.

For clarity of the proofs, we do not make any attempts to optimize the constants.

2 Strongly crossing edges in the plane

In this section we will prove Theorems 1.2 and 1.3. Recall that a *geometric graph* is a graph drawn in the plane with vertices represented by points and edges by straight line segments connecting the corresponding pairs. Recently Ackerman [1] showed the following.

Lemma 2.1. *Let $G = (V, E)$ be an n-vertex geometric graph in the plane with no four pairwise crossing edges. Then $|E(G)| \leq O(n)$.*

□

We note that Lemma 2.1 holds for topological graphs. Before we give the proofs, we will introduce some terminology. Consider a family $\mathcal{S} = \{s_1, ..., s_k\}$ of pairwise crossing segments in the plane, and let $\mathcal{L} = \{l_1, ..., l_k\}$ be a family of lines such that l_i is the line supported by segment s_i. Recall that the *level* of a point $x \in \cup\mathcal{L}$ is defined as the number of lines of $\mathcal{L}$ lying strictly below x. We

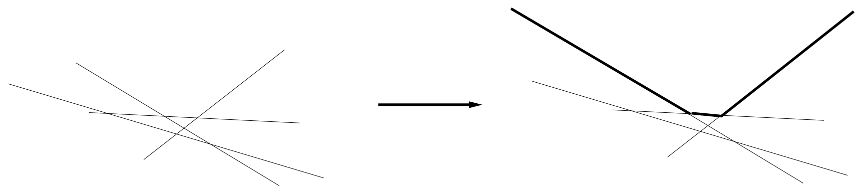


Figure 2: The top level of four pairwise crossing segments is drawn thick.

define the *top level of* $\mathcal{L}$ as the closure of the set of points in $\cup\mathcal{L}$ with level $k-1$. We define the *top level of* $\mathcal{S}$ to be the top level of $\mathcal{L}$. See Figure 2. Notice that L is a (not strictly) convex function.

For each edge t in a geometric 3-hypergraph in the plane, we define its *base* as the side with the longest x-projection. We define the other two sides of t as its *left* and *right* side. See Figure 2. Notice that every edge in a geometric 3-hypergraph is incident to a vertex that lies strictly above or below its base. We are now ready to prove Theorem 1.2.

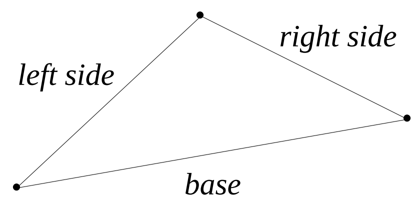


Figure 3: The base, left side, and right side.

Proof of Theorem 1.2. Let $H = (V, E)$ be an n-vertex geometric 3-hypergraph in the plane with no three strongly crossing edges. We can assume that $|E(H)| \geq 20n^2$ (since otherwise we would be done) and at most $|E(H)|/2$ edges in H are incident to a vertex that lies strictly below its base. We will discard all such edges, leaving us with at least $|E(H)|/2$ edges left. Let E_{uv} be the set of edges in H with base uv. We discard all sets E_{uv} for which $|E_{uv}| \leq |E(H)|/(2n^2)$. Since we have thrown away at most $|E(H)|/4$ edges in this process, we have at least $|E(H)|/4$ edges left. Therefore $|E_{uv}| = 0$ or $|E_{uv}| \geq |E(H)|/(2n^2) \geq 10$.

Now let $G_v = (V, E)$ denote the geometric graph with $V(G_v) = V(H)$ and $xy \in E(G_v)$ if $conv(x \cup y \cup v) \in E(H)$ with base xy.

Observation 2.2. *G_v does not contain four pairwise crossing edges (bases).*

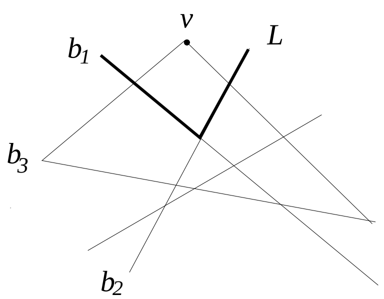
Proof. For sake of contradiction, suppose G_v contains four pairwise crossing edges $b_1, b_2, b_3, b_4 \in E(G_v)$. Then v lies above b_i for all i. Let L denote the top level of the arrangement $\mathcal{S} = \{b_1, b_2, b_3, b_4\}$. Now the proof falls into three cases.

Case 1. Suppose L intersects exactly two members of $\mathcal{S}$, say bases b_1 and b_2 (in order from left to right along L). Let p be the intersection point of b_1 and b_2. Then the vertical line through p must intersect b_3 below p. Moreover, since segments b_1 and b_3 cross, v and the right-endpoint of b_3 must lie on the same half-plane generated by the line supported by b_1. Likewise, v and the left-endpoint of b_3 must lie on the same half-plane generated by the line supported by b_2. Therefore

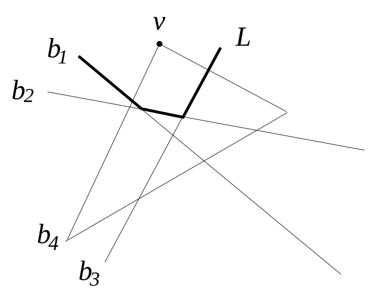
$p \in conv(v \cup b_3)$. See Figure 4(a). Since $|E_{b_1}|, |E_{b_2}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $conv(v \cup b_3), conv(x \cup b_1), conv(y \cup b_2)$ are three (vertex disjoint) strongly crossing edges in H and we have a contradiction.

Case 2. Suppose L intersects exactly three members of $\mathcal{S}$, say bases b_1, b_2, b_3 (in order from left to right along L). Now b_4 must intersect b_2 either to the left or right of $b_2 \cap L$. Without loss of generality, we can assume that b_4 intersects b_2 to the right of $b_2 \cap L$. Let p be the intersection point of segments b_2 and b_3. By the same argument as above, $p \in conv(v \cup b_4)$. See Figure 4(b). Since $|E_{b_1}|, |E_{b_2}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $conv(v \cup b_4), conv(x \cup b_1), conv(y \cup b_2)$ are three strongly crossing edges in H and we have a contradiction.

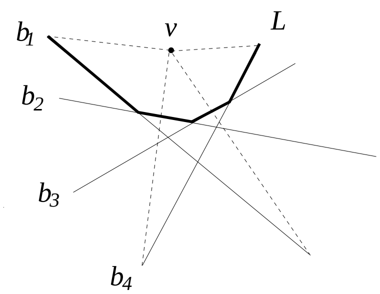
Case 3. Suppose L intersects b_1, b_2, b_3, b_4 in order from left to right along L. Let p be the intersection point of segments b_2 and b_3, and let l be the vertical line through v. Since the right endpoint of b_4 lies to the right of l, and the left endpoint of b_1 lies to the left of l, we have $p \in conv(v \cup b_1) \cup conv(v \cup b_4)$. Therefore, either $conv(v \cup b_1)$ or $conv(v \cup b_4)$ (say $conv(v \cup b_1)$) contains p. See Figure 4(c). Since $|E_{b_2}|, |E_{b_3}| \geq 10$, there exists vertices $x, y \in V(H)$ such that $conv(v \cup b_1), conv(x \cup b_2), conv(y \cup b_3)$ are three strongly crossing edges in H and we have a contradiction.



(a) Case 1.



(b) Case 2.



(c) Case 3.

Figure 4: Three cases.

□

Therefore by Lemma 2.1, $|E(G_v)| \leq O(n)$ for every vertex $v \in V(H)$. Hence

$$\frac{|E(H)|}{4} \leq \sum_{v \in V(H)} |E(G_v)| = O(n^2),$$

which implies $|E(H)| = O(n^2)$.

□

Before we prove Theorem 1.3, we will need the following lemma due to Valtr [18].

Lemma 2.3. *Let $G = (V, E)$ be an n-vertex geometric graph in the plane such that all of the edges in G intersect the y-axis. If G does not contain k pairwise crossing edges, then $|E(G)| \leq c_k n$ where c_k depends only on k.*

□

Proof of Theorem 1.3. Let H be an n-vertex geometric 3-hypergraph in the plane with no k strongly crossing edges for $k \geq 4$. Just as before, we can assume at most $|E(H)|/2$ of the edges in

H are incident to a vertex that lies strictly below its base. We discard all such edges, leaving us with at least $|E(H)|/2$ edges left in H. Now we make the following observation.

Observation 2.4. *Suppose $b_1, ..., b_k$ are k pairwise crossing bases and $v_1, ..., v_k \in V(H)$ such that $conv(v_i \cup b_j) \in E(H)$ with base b_j for all i, j. Then H contains k strongly crossing edges.*

Proof. Let L denote the top level of the segment arrangement $\mathcal{S} = \{b_1, ..., b_k\}$ and assume that $b_1, ..., b_k$ are ordered by increasing slopes. See Figure 5.

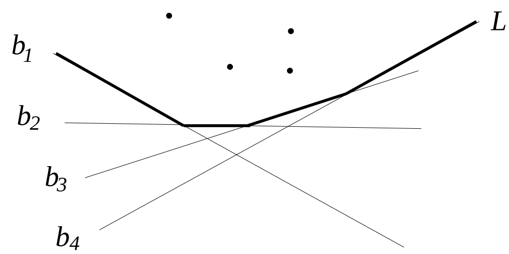


Figure 5: Arrangement of b_1, b_2, b_3, b_4.

Now we define edges $t_1, t_2, ..., t_k \in E(H)$ as follows. Among the k edges $conv(b_1 \cup v_1), conv(b_1 \cup v_2), ..., conv(b_1 \cup v_k) \in E(H)$, (with slight abuse of notation) let $t_1 = conv(b_1 \cup v_1)$ be the edge whose right side has the rightmost intersection with L. Then among the $k-1$ edges $conv(b_2 \cup v_2), conv(b_2 \cup v_3), ..., conv(b_2 \cup v_k)$, (again with slight abuse of notation) let $t_2 = conv(b_2 \cup v_2)$ be the edge whose right side has the rightmost intersection with L. We continue this procedure until we have k edges $t_1, t_2, ..., t_k$. Clearly these k edges are vertex disjoint.

Now notice that $(t_i \cap L) \cap (t_j \cap L) \neq \emptyset$ for all pairs i, j. Indeed for sake of contradiction, suppose there exists two edges t_i and t_j for $i < j$ such that either $t_i \cap L$ lies completely to the left of $t_j \cap L$ or vice versa. See Figure 6.

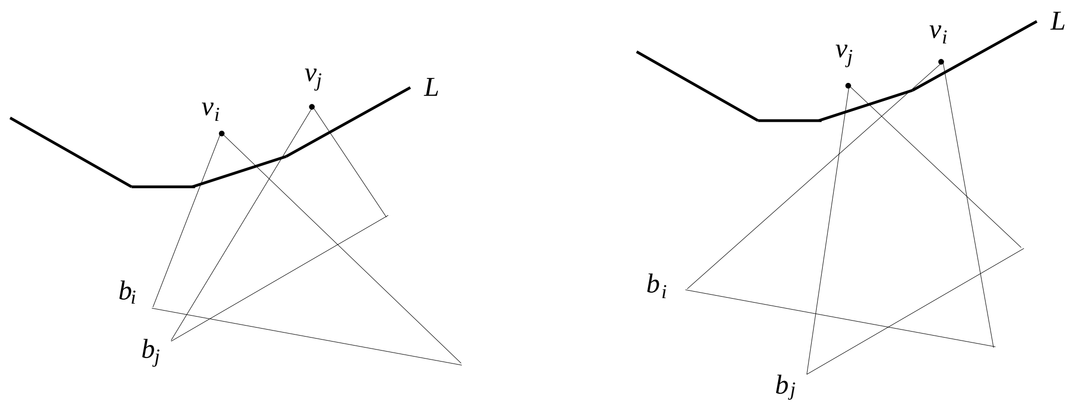


Figure 6: Assume $(t_i \cap L) \cap (t_j \cap L) = \emptyset$.

Case 1. Suppose $t_i \cap L$ lies completely to the left of $t_j \cap L$. Then the vertical line through v_j intersects the right side of t_i below v_j. Therefore the right side of $conv(b_i \cup v_j)$ intersects L more to the right than the right side of $t_i = conv(b_i \cup v_i)$ does. This contradicts the definition of t_i and t_j.

Case 2. Suppose $t_i \cap L$ lies completely to the right of $t_j \cap L$. Then there exists a base b_s that has a point p on L between $t_i \cap L$ and $t_j \cap L$. Base b_s must

1. lie below v_i and v_j,
2. cross b_i and b_j, and
3. contain point p.

However this impossible by the following argument. Let l be the vertical line through p. Clearly l intersects b_i and b_j. Since b_s lies below v_i and v_j, b_s must intersect b_j to the left of l, and intersect b_i to the right of l. Since b_s intersects b_j to the left of l, the slope of b_s must be greater than the slope of b_j. However since the slope of b_i is less than the slope of b_j, this implies that b_s cannot intersect b_i to the right of l. Hence we have a contradiction.

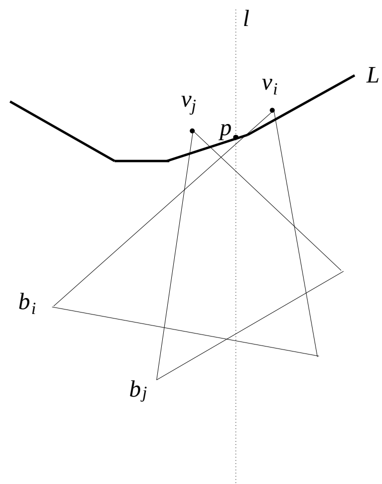


Figure 7: Case 2.

Since $(t_i \cap L) \cap (t_j \cap L) \neq \emptyset$ for every $i, j \in \{1, 2, .., k\}$, by Helly's Theorem [6] $t_1, ..., t_k$ has a nonempty intersection on L.

□

Notice that no k points in $V(H)$ have $c_k n$ bases in common. Indeed, otherwise the vertical line through any of these k points would intersect all $c_k n$ bases, and by Lemma 2.3 there would be k pairwise crossing bases. By Observation 2.4, we would have k strongly crossing edges.

Now let $G = (A \cup B, E)$ be a bipartite graph where $A = V(H)$ and $B = V^2(H)$, such that $(v, xy) \in E(G)$ if $conv(x \cup y \cup v) \in E(H)$ with base xy. Since G does not contain $K_{k,c_k n}$ as a subgraph, we can use the following well known result of Kővári, Sós, Turán [10].

Theorem 2.5. *If $G = (A \cup B, E)$ is a bipartite graph with $|A| = n$ and $|B| = m$ containing no subgraph $K_{r,s}$ with the r vertices in A and the s vertices in B, then*

$$|E(G)| \leq (s-1)^{1/r} n m^{1-1/r} + (r-1)m.$$

By plugging in the values $m = n^2, r = k, s = c_k n$ into Theorem 2.5, we obtain

$$\frac{|E(H)|}{2} \leq |E(G)| \leq O\left(n^{3-\frac{1}{k}}\right).$$

Hence

$$|E(H)| \leq O\left(n^{3-\frac{1}{k}}\right).$$

□

2.1 Convex geometric 3-hypergraphs

In the case when the vertices are in convex position in the plane, extremal problems on geometric 3-hypergraphs become easier due to the linear ordering of its vertices. The proof of Observation 2.4 can be copied almost verbatim to conclude the following.

Observation 2.6. *Let $H = (V, E)$ be a geometric 3-hypergraph in the plane with vertices in convex position. Suppose H contains k edges of the form $t_i = conv(x_i \cup y_i \cup z_i)$, such that the vertices $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_k)$ appear in clockwise order along the boundary of their convex hull. Then $t_1, ..., t_k$ are k strongly crossing edges.*

□

Marcus and Klazar [9] extended the Marcus-Tardos theorem [13] by showing that the number of 1-entries in a r-dimensional $(0, 1)$-matrix with side length n which avoids an r-dimensional permutation matrix is $O(n^{r-1})$. As pointed out by Marcus and Klazar, it is not difficult to modify their proof to obtain an $O(n^{r-1})$ bound on the number of edges in an ordered n-vertex r-uniform hypergraph that does not contain a fixed ordered matching. Hence by Observation 2.6, we can conclude the following.

Theorem 2.7. *Let $H = (V, E)$ be a geometric 3-hypergraph in the plane with vertices in convex position. If H does not contain k strongly crossing edges, then $|E(H)| \leq c_k n^2$ where c_k is a constant that depends only on k.*

□

3 Disjoint edges in 3-space

In this section, we will prove Theorem 1.5. Recall that two edges in a geometric graph are *parallel* if they are the opposite edges of a convex quadrilateral. Katchalski and Last [7] and Pinchasi [16] showed that all n-vertex geometric graphs with more than $2n - 2$ edges contain two parallel edges. By following Pinchasi's argument almost verbatim, one can prove the following.

Lemma 3.1. *Let G be a graph drawn on the unit sphere S with vertices represented as points such that no three lie on a great circle, and edges $uv \in E(G)$ are drawn as arcs along the great circle containing points u and v of length less than π (the shorter arc). We say that edges $e_1, e_2 \in E(G)$ are avoiding if the great circle supported by e_1 is disjoint to e_2, and the great circle supported by e_2 is disjoint from e_1. If $|E(G)| > 2n - 2$, then G contains two avoiding edges.*

□

Proof of Theorem 1.5. Let $H = (V, E)$ be an n-vertex geometric 3-hypergraph in 3-space with no two disjoint edges. Fix a pair of vertices $u, v \in V(H)$, and just consider the edges $E_{uv} = \{t \in E(H) : u, v \text{ are vertices of } t\}$. We color $t \in E_{uv}$ *red* if all of the members of E_{uv} lie in one of the closed half-spaces generated by the plane supported by t. Notice that there are at most two red edges in E_{uv}. Repeat this procedure for each pair of vertices, which will leave us with at most n^2 red edges in the end. Color the remaining edges blue, and let $d_b(v)$ denote the number of blue edges incident to v. Then we have

$$\sum_{v \in V(H)} d_b(v) \geq 3E(H) - 3n^2.$$

Therefore, there exists a vertex v incident to at least $(3|E(H)| - 3n^2)/n$ blue edges. Now consider a small 2-dimensional sphere S^2 centered at v. Then the intersection of S^2 and the blue edges incident to v forms a graph G with at most n vertices and at least $(3E(H) - 3n^2)/n$ edges.

If $(3|E(H)| - 3n^2)/n > 2n - 2$, then by Lemma 3.1 we know that G contains two avoiding edges xy and wz. Let h be the plane supported by the blue edge $conv(w \cup z \cup v) \in E(H)$. Then the blue edge $conv(x \cup y \cup v)$ must lie in one of the closed half-spaces generated by the plane h. Since $conv(w \cup z \cup v)$ is blue, there must be a red edge $conv(w \cup z \cup p)$ such that h separates it from $conv(x \cup y \cup v)$. Hence $conv(x \cup y \cup v)$ and $conv(w \cup z \cup p)$ are disjoint and we have a contradiction. See Figure 8. Therefore $(3|E(H)| - 3n^2)/n \leq 2n - 2$, which implies $|E(H)| \leq O(n^2)$.

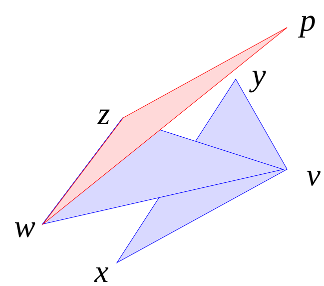


Figure 8: Disjoint edges $conv(w \cup z \cup p)$ and $conv(x \cup y \cup v)$.

□

4 Remarks

By applying the Abstract Crossing Lemma (see [19]) to Theorem 1.2, every n-vertex geometric 3-hypergraph H in the plane has either $O(n^2)$ edges or $\Omega(|E(H)|^7/n^{12})$ triples that have a point in common. In the latter case, by the fractional Helly theorem [8] this implies one can always find a point inside at least $\Omega(|E(H)|^5/n^{12})$ edges of H. However, this is not as strong as the

$$\Omega\left(\frac{|E(H)|^3}{n^6 \log^2 n}\right)$$

bound obtained by Nivasch and Sharir [14].

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